CONFIDENCE INTERVALS FOR SIGNAL TO NOISE RATIO

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Confidence Interval estimates for signal to noise ratio based on sample correlation coefficient are provided herein. Previously, Barshad and Rockmore [1] provided only *point estimate* for signal to noise based on the sample correlation coefficient. Thus the present paper extends the work on point estimate to one on *interval estimators*.

SIGNAL TO NOISE RATIO $\alpha=\rho/1-\rho$

Under the usual two channel Gaussoan set up, let

$$
\begin{aligned}
x(t) &= s(t)+n_1(t) \\
y(t) &= s(t)+n_2(t)
\end{aligned}
\qquad \ldots(1)
$$

represent the two familiar wave forms. Further, let $R=(x_1,\ldots, x_N, y_1,\ldots y_N)$ denote the random vectors of time samples such that R is jointly Gaussian with mean zero and variance covariance matrix, Λ, (P_N=noise power)

$$
\Lambda = P_N \begin{bmatrix} (1+\alpha)\ I & \alpha\ I \\ \alpha\ I & (1+\alpha)\ I \end{bmatrix} \qquad \ldots(2)
$$

α being the well known signal to noisc ratio, and I is the $N\times N$ unit matrix. It is well known from information theory that the signal to noise ratio α can be expressed as $\alpha=\rho/1-\rho$, ρ being the product moment correlation coefficient in the bivariate normal distribution. Barshad and Rockmore provided only *point estimate* for the signal to noise ratio $\hat{\alpha}=r/1-r$, $\hat{\alpha}$ being based on the sample correlation r. They demonstrated that the point estimate is unbased and asymptotically efficient.

CONFIDENCE INTERVAL ESTIMATES FOR $\alpha=\rho/1-\rho$

Let the estimate of α be $\hat{\alpha}=r/1-r$.

Then it is easily verified that

$$\text{Var. } (\hat{\alpha}) \approx \frac{(1+\rho)^2}{\rho^{2}N} \simeq \frac{(1+2\alpha)^2}{N} \text{ to order } N^{-1}. \quad \ldots(3)$$

Using the familiar variance stabilising results in theoretical statististics the transformation

$$\int \frac{d\hat{\alpha}}{(1+2\hat{\alpha})} = \tfrac{1}{2} \log (1+2\hat{\alpha}) \quad \ldots(4)$$

has a distribution asymptotically normal as $n\to\infty$ with mean $\frac{1}{2}\log(1+2\alpha)$ and variance $1/N-3$ *independent* of α. But this transformation (4) is the familiar transformation of Fisher (which result was used by Barshad and Rockmore without a formal proof). So assuming

$$\sqrt{N-3} \left[\frac{1}{2}\log (1+\hat{\alpha}) - \frac{1}{2} \log (1+2\alpha) \right] \quad \ldots(5)$$

is standardised normal, the $100(1-\beta)$ symmetric confidence intervals based on this transformation are provided by

$$P\left[\frac{1}{2}\left\{ e^{-A}(1+2\hat{\alpha})-1 \right\} \leqslant \alpha \leqslant \frac{1}{2}\left\{ e^{A}(1+2\hat{\alpha})-1 \right\} \right]$$
$$=1-\beta; \quad \ldots(6)$$
$$A=2z_{\beta/2}/\sqrt{N-3}.$$

Similarly symmetric confidence intervals based on the arc-sine transformations of Sankar an [2]

$$V=\sin^{-1}\left(\frac{r-\rho}{1-\rho r}\right)$$

and of Nair's transformation

$$U=\frac{r-\rho}{1 \quad \rho r}$$

(vide Sankaran [2], [3], [4], [5] are provided by

$$P\left[\frac{(1-B)\hat{\alpha}-B}{1+B} \leqslant \alpha \leqslant \frac{(1+B)\hat{\alpha}+B}{1-B}\right]=1-\beta\,;$$

$$B=\sin\left(\frac{v_{\beta/2}}{\sqrt{N-2}}\right) \qquad \ldots(7)$$

and

$$P\left[\frac{(1-C)\hat{\alpha}-C}{1+C} \leqslant \alpha \leqslant \frac{(1+C)\hat{\alpha}+C}{1-C}\right]=1-\beta\,;$$

$$C=\frac{u_{\beta/2}}{\sqrt{N-1}} \qquad \ldots(8)$$

where $z_{\beta/2}$, $v_{\beta/2}$ and $u_{\beta/2}$ are the usual normal probabilities cut off such that

$$\int_a^\infty \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}dt=\beta/2. \qquad \ldots(8a)$$

In practice, the shortest of the three confidence intervals (6), (7) and (8) is to be preferred. The theoretical justification is provided by Sankaran [3] for V and U in preference to Z.

Tables I and II provide 95% and 90% confidence limits for the signal to noise ratio α based on V, Z and U transformations for the particular sample size n=25. From Table I we conclude that *confidence intervals based on V and U are comparable* whereas those based on U are somewhat wider. From Table II we see that confidence intervals based on all the three transformations are compareable. In the preparation of the tables we have been assisted by Sri S. Pearson and S. Shrikanthan, 95% and 90% confidence charts for sample sizes n=25, 30, 40, 50 and 100 are in preparation and will be issued later as separates. Earlier, 95% and 90% confidence charts for parential correlation based on V and U were prepared and issued by Sankaran (1973).

95% Confidence Limits for Signal-Noise Ratio for N=25 and Various $\hat{\alpha}$

	N = 1,00,000	10,000	1,000	100	10	
	L ; *U*	*L* ; *U*	*L* ; *U*	*L* ; *U*	*L* ; *U*	
V	43123 ; 231896	4312.4 ; 23190.2	430.94 ; 2319.61	42.839 ; 232.555	4.0280 ; 23.8485	
Z	43400 ; 230401	4339.7 ; 23040.7	433.72 ; 2304.65	43.117 ; 231.052	4.5570 ; 23.6920	
U	42025 ; 237953	4202.2 ; 23795.9	421.41 ; 2380.21	41.736 ; 238.642	3.9127 ; 24.4846	
	7	4	1	3/4	1/2	
V	2.7342 ; 16.8192	1.4405 ; 9.9353	.1469 ; 2.9784	0.0390 ; 2.3907	—.0688 ; 1.8228	
Z	2.7550 ; 16.7800	1.5030 ; 9.8680	.1510 ; 2.9560	0.0425 ; 2.3800	—.0660 ; 1.8040	
U	2.6306 ; 17.3461	1.3911 ; 10.2078	.1304 ; 3.0693	0.0253 ; 2.4744	—.0752 ; 1.8801	
	1/3	1/4	1/5	1/6	1/7	1/8
V	—.1406 ; 1.4324	—.1766 ; 1.2390	—.1981 ; 1.1233	—.2124 ; 1.0509	—.2228 ; 0.9908	—.2305 ; .9491
Z	—.1384 ; 1.4200	—.1745 ; 1.2280	—.1962 ; 1.1128	—.2107 ; 1.0360	—.2210 ; 0.9812	—.2280 ; .9400
U	—.1498 ; 1.4829	—.1848 ; 1.2846	—.2058 ; 1.1657	—.2198 ; 1.0863	—.2298 ; 1.0297	—.2381 ; .9872
	1/9	1/10	0	—1/10	—1/19	—1/8
V	—.2392 ; .9171	—.2413 ; .8914	—.2844 ; .6595	—.3275 ; .4276	—.3323 ; .4018	—.3383 ; .3696
Z	—.2348 ; .9080	—.2396 ; .8826	—.2830 ; .6520	— 3264 ; .4212	—.3312 ; .3960	—.3373 ; .3640
U	—.2432 ; .9542	—.2478 ; .9277	—.2899 ; .6898	—.3319 ; .4518	—.3364 ; .4254	—.3438 ; .3923
	—1/7	—1/6	—1/5	—1/14	—1/3	$\hat{\alpha}=-1/2$ $\alpha=-1/2$
V	—.3460 ; 3219	—.3563 ; .2730	—.3722 ; .1990	—.3922 ; .0797	—.4281 ; —.1135	
Z	—.3450 ; 3229	—.3554 ; .2680	—.3698 ; .1912	—.3915 ; .0760	—.4277 ; —.1160	
U	—.3499 ; 3498	—.3599 ; .2932	—.3736 ; .2139	—.3956 ; .0949	—.4297 ; —.1035	

L=Lower limit ; *U*=Upper limit

90% Confidence Limits for Signal-To-Noise Ratio for N=25 and Various $\hat{\alpha}$

	N 1,00,000	10,000	1,000	100	10	
	L ; *U*	*L* ; *U*	*L* ; *U*	*L* ; *U*	*L* ; *U*	
V	49667 ; 201341	4966.4 ; 20134.6	496.42 ; 2013.92	49.415 ; 201.848	4.7150 ; 20.6407	
Z	49509 ; 201411	4958.7 ; 20141.5	495.65 ; 2014.61	49.338 ; 201.917	4.7070 ; 20.6481	
U	49712 ; 201151	4971.0 ; 20116.5	496.87 ; 2012.10	49.460 ; 201.664	4.7198 ; 20.6217	
	7	4	1	3/4	1/2	
	L ; *U*	*L* ; *U*	*L* ; *U*	*L* ; *U*	*L* ; *U*	
V	3.2252 ; 14.6006	1.7328 ; 8.5602	0.2450 ; 2.5201	0.1209 ; 2.0167	—.0033 ; 1.5135	
Z	3.2193 ; 14.6058	1 6816 ; 8.5625	0.2439 ; 2.5212	0.1169 ; 2.0172	—.0041 ; 1.5141	
U	3.2290 ; 14.5870	1.7370 ; 8,5522	0.2457 ; 2.5174	0.1214 ; 2,0146	—.0029 ; 1.5104	
	1/3	1/4	1/5	1/6	1/7	1/8
V	—.0861 ; 1.1778	—.1275 ; 1.0101	—.1523 ; .9095	—.1689 ; .8423	—.1807 ; .7943	—.1896 ; .7584
Z	—.0863 ; 1.1784	—.1280 ; 1.0106	—.1529 ; .9099	—.1694 ; .8428	—.1807 ; .7948	—.1901 ; .7588
U	—.0854 ; 1.1763	—.1272 ; 1.0087	—.1520 ; .9081	—.1686 ; .8411	—.1804 ; .7932	—.1893 ; .7573
	1/9	1/10	0	—1/10	—1/9	—1/8
V	—.1965 ; .7304	—.2020 ; .7080	—.2518 ; .5067	—.3013 ; .3054	—.3068 ; 2830	—.3137 ; .2565
Z	—.1970 ; .7308	—.2025 ; .7084	—.2521 ; .5076	—.3017 ; .3057	—.3072 ; 2833	—.3140 ; .2553
U	—.1962 ; .7293	—.2017 ; .7070	—.2514 ; .5058	—.3012 ; .3046	—.3067 ; 2823	—.3136 ; .2543
	—1/7	—1/6	—1/5	—1/4	—1/3	
V	—.3226 ; .2191	—.3344 ; .1711	—.3510 ; .1040	—.3758 ; .0033	—.4172 ; —.1642	$\hat{\alpha}=-1/2$
Z	—.3229 ; .2193	—.3347 ; .1709	—.3512 ; .1043	—.3760 ; .0035	—.4174 ; —.1643	$\alpha=-1/2$
U	—.3225 : .2184	—.3343 ; .1720	—.3509 ; .1035	—.3757 ; .0029	—.4172 ; —.1647	

L=Lower limit ; U= Upper limit

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